Free Vibration Analysis of Rotating Functionally Graded Beams Using the p-Version of the Finite Element Method

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Abstract: *The functionally graded materials (FGMs) are inhomogeneous composite materials where the properties of FGM constituents vary gradually and smoothly. The smooth variation of the material properties overcomes the adverse effect of the laminate and sandwich composites structure such as the delamination mode of failure caused by the large interlaminar stresses. In this study, the free vibration of the lead-lag motion of a rotating functionally graded beam (RFGB) is investigated. From Hamilton's principle, the linear partial differential equations are derived for coupled stretching and bending motion. The governing equations based on Euler–Bernoulli beam theory accounts for centrifugal forces field, the centripetal acceleration and the gyroscopic effect. A p-version of the finite element method in conjunction with the modeling dynamic method using the arc-length stretch deformation is applied to find natural frequencies and modes shape of the cantilever beam. The displacements are expressed as the combination of the in-plane and out-of-plane shape functions, enriched with trigonometric hierarchical shape functions used generally to give additional degrees of freedom (dof) to the interior of the element. The convergence properties of the rotating beam Fourier p-element is examined, the results are compared with those of the literature where excellent agreements are observed. The influence of angular speed, Young's modulus ratio and power-law exponent on the natural frequencies and mode shapes is investigated. The tuned rotating speeds at which the beam will vibrate violently are determined for stainless steel–silicon nitride RFGB versus the power-law exponent.*

Keywords: *rotating functionally graded beam, Fourier p-element, lead-lag free vibration, gyroscopic effect, hybrid displacements, tuned rotating speed*

1. Introduction

There are many examples in mechanics which can be modeled as rotating cantilever beams, such as turbine blades, turbo-engine blades, helicopter blades, robot manipulators and wind blades. Knowledge of the vibrational behavior of these structures helps to avoid resonance and instability problems during their operational life. The functionally graded materials (FGMs) are composite materials where the microstructures are locally varied. The composition and the volume fraction of FGM constituents vary gradually, giving a non uniform microstructure with continuously graded macro properties. For instance, one face of a structural component may be an engineering ceramic that can resist severe thermal loading, and the other face may be a metal to maintain structural rigidity and toughness. The smooth variation of the material properties in the body overcomes the adverse effect of the laminate and sandwich composites structure such as the delamination mode of failure caused by the large interlaminar stresses. Details of design, processing and applications of FGMs can be found in Koizumi (1997) and Miyamoto and al. (1999).

The research work related to the modeling and behavior of rotating functionally graded beams (RFGB) made of FGMs has been limited to a few papers. Librescu and his co-workers (Oh *& al.*2003, Librescu *& al.*2005, Librescu *& al.*2008) are the first ones dealing with a rotating blade made of FGMs. The blade, mounted on a rigid hub, is modeled as a thin-walled beam that incorporates the warping restraint and the pretwist effects. Librescu and his co-workers have treated the problems of free vibration and thermoelastic modeling of turbomachinery thin-walled rotating blades made of FGMs and operating in a high temperature field. Fazelzadeh and Hosseini (2007) investigate the aerothermoelastic behavior of supersonic rotating thin-walled beams made

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$$I_A\left[v_{0,tt}+2\Omega S_{,t}-\Omega^2 v_0+\dot{\Omega}\,S\right]+I_B S_{,xtt}-I_A\Omega^2\left[\left(v_{0,x}\right)\left(\frac{1}{2}(L^2-x^2)\right)\right]_{,x}+I_D v_{0,xxtt}+D_{xx}v_{0,xxxx}-B_{xx}S_{,xxx}=I_A\dot{\Omega}\,x \tag{8}$$

The related boundary conditions are given by

$$S=v_0=v_{0,x}=0 \text{ at } x=0 \tag{9}$$

$$S_{,x}=v_{0,xx}=v_{0,xxx}=0 \text{ at } x=L \tag{10}$$

3. Fourier p-element Formulation

In the *p*-version of the FEM, the accuracy of the approximation is improved by increasing the number of shape functions over the element, keeping the mesh constant. The RFGB is modeled as just one finite element and the number of trigonometric terms is varied. For irregular geometries more elements can be used. The nodal dof of the element at each node are the transverse displacement v_0, the slope $v_{0,x}$, the curvature $v_{0,xx}$ and the stretching displacement S.

The local and non-dimensional coordinates are related by

$$\xi=\frac{x}{L}, \qquad 0\le\xi\le 1 \tag{11}$$

The displacement vector formed by the hybrid variables S and v_0 may be expressed as the combination of the in-plane and out-of plane hierarchical shape functions and can be written as

$$\begin{cases} S(\xi,t)=\displaystyle\sum_{k=1}^{M_S} S_k(t)f_k(\xi) \\ v_0(\xi,t)=\displaystyle\sum_{k=1}^{M_v} Y_k(t)g_k(\xi) \end{cases} \tag{12}$$

After performing the conventional steps of variational calculus and applying Galerkin's method to Eqs. (7) and (8), the system of linear algebraic equations of motion for free vibration of a RFGB can be obtained. The system is a two coupled linear equations of motion. This system which defines the lead-lag vibration is given as follows

$$\sum_{r=1}^{M_S+M_v} M_{r,s}\,\ddot{q}_s+2\Omega G_{r,s}\dot{q}_s+\left[K_{r,s}+\Omega^2\left(R_{r,s}-M^*_{r,s}\right)+\dot{\Omega}G_{r,s}\right]q_s=0,$$

$$s=1,2,3,\dots,M_s+M_v \tag{13}$$

Where $\dot{\Omega}$ is the angular acceleration, $M_{r,s}$, $K_{r,s}$ are the coefficients of the conventional hierarchical finite element mass and stiffness matrix, $2\Omega G_{r,s}$ are the coefficients of the gyroscopic matrix accounts for Coriolis effects, $\Omega^2 R_{r,s}$ are the elements of the additional stiffness matrix caused by the centrifugal effect, while $\Omega^2 M^*_{r,s}$ are the elements of the additional stiffness matrix caused by the centripetal acceleration.

4. Convergence Study and Comparison

In this section, solution accuracy and convergence studies of the present formulation are carried out. Vibration study of homogeneous and functionally graded beams with and without rotation is considered here. The results obtained by the proposed approach are compared with available literature. The frequency parameter and other dimensionless parameters are introduced

$$\omega = \sqrt{\frac{\rho_l A L^4}{E_l I}}\,\varpi, \mu = \sqrt{\frac{\rho_l A L^4}{E_l I}}\,\Omega, \lambda = \sqrt{\frac{A L^2}{I}}, \rho_{\text{ratio}} = \frac{\rho_{\boldsymbol{u}}}{\rho_{\boldsymbol{l}}}, E_{ratio} = \frac{E_u}{E_l} \tag{14}$$

where ω, μ and λ are the frequency parameter, the rotating speed parameter, the slenderness ratio, Young's modulus ratio and mass density ratio, respectively.

In order to see the manner of convergence, the beam is considered in the stationary state ($\mu = 0$) and slender ($\lambda = 70$), only one element is used in the discretization and the number of hierarchical terms is varied. Results of the first four lowest bending modes of a cantilever homogeneous beam are illustrated in Table 1, along with the exact solutions, the finite element method (Chung and Yoo 2002), and the solutions from the hierarchical finite element method where an usual beam element with two degrees of freedom (transverse displacementw and the slope $v_{,x}$) by node is used (Hamza-Cherif 2005).The number of hierarchical terms used in this study is 8 and the corresponding number of system degrees of freedom is 12. The number of dof is 26 in (Hamza-Cherif 2005) and 198 in (Chung and Yoo 2002). Table 1 clearly shows that convergence from above to the exact values occurs as the number of trigonometric hierarchical terms is increased and highly accurate solutions are obtained despite the use of few hierarchical terms. The proposed trigonometric hierarchical finite element with transverse, slope and curvature dof solutions are more accurate than the trigonometric hierarchical finite element with only transverse and slope solutions (Hamza-Cherif 2005). This would save about 60 % and 90 % of the system dof than the solutions proposed in (Hamza-Cherif 2005) and (Chung and Yoo 2002), respectively. An upper-bound solution to the exact values, uniform and monotonic convergence is guaranteed.

TABLE I: Convergence of the lowest frequency parameters ω of the cantilever homogeneous beam as a function of the number of the hierarchical terms when μ=0,λ=70,E_{ratio}=1 and ρ_{ratio}=1.

Method	dof(N_v)	Bending modes			
		1st	2nd	3rd	4th
Converged solution	4(0)	3.5160	22.1578	63.3466	281.5963
	6(2)	3.5160	22.0346	61.7017	127.5941
	8 (4)	3.5160	22.0345	61.6973	120.9214
	10 (6)	3.5160	22.0345	61.6972	120.9023
	12 (8)	3.5160	22.0345	61.6972	120.9019
HFEM (Hamza-Cherif 2005)	26	3.5160	22.0345	61.6972	120.9019
FEM (Chung and Yoo 2002)	198	3.5160	22.0345	61.6972	120.9019
Exact		3.5160	22.0345	61.6972	120.9019

Table 2 gives the results of the first five frequency parameters in the case of RFGB with the rotating speed parameter $\mu = 10$. The parameters of the beam are as follows: $\lambda = 70$, $\rho_{\text{ratio}} = 1 E_{ratio} = 4$ and $n = 0.5$. The number of shape functions is increased from 2 to 20 for M_v and from 2 to 24 for M_s, the relative decrease of the frequency parameter is less than $2\ 10^{-4}$%, showing relatively rapid convergence characteristics. By comparing the number of shape functions in Tables 1 and 2, it is clear that the number required in rotating beams is greater than required in stationary state because the convergence characteristics is affected by centrifugal stiffening.

Comparison between the proposed model and the previously published works in the case of a simply supported functionally graded beam is presented in Tables 3-5. The width of the beam is taken equal to 0.4 m, where the length L is 20 m. The frequency parameter used here is

$$\omega^2 = \sqrt{\frac{\rho_l A L^4}{E_l I}}\,\varpi \tag{15}$$

TABLE 2: Convergence of the first frequency parameter ω of the cantilever FG beam as a function of the number of the hierarchical terms when μ=10,λ=70,E_{ratio}=4,ρ_{ratio}=1 and n=0.5.

M_v (M_S)	Bending modes				
	1st	2nd	3rd	4th	5th
0 (4)	7.0145	43.1574	111.2990	191.2905	448.8573
2 (6)	7.0110	43.0641	108.8831	191.2222	215.4220
4 (8)	7.0107	43.0623	108.8498	191.1871	205.1481
6 (10)	7.0106	43.0615	108.8453	191.1844	205.1175
8 (12)	7.0105	43.0612	108.8432	191.1832	205.1109
10 (14)	7.0105	43.0610	108.8422	191.1826	205.1075
12 (16)	7.0105	43.0609	108.8416	191.1822	205.1055
14(18)	7.0105	43.0608	108.8412	191.1820	205.1043
16(20)	7.0105	43.0607	108.8409	191.1818	205.1035
18(22)	7.0105	43.0607	108.8408	191.1817	205.1030
20(24)	7.0105	43.0607	108.8407	191.1817	205.1026
Converged solution	7.0105	43.0607	108.8407	191.1817	205.1026

5. Results

Frequencies of stainless steel–silicon nitride RFGB

Free vibration analysis of RFGB is considered in this section. The FGM is composed of stainless steel (SS) and silicon nitride (SN) and its properties changes through the thickness of the beam. Therefore, the top surface of the beam is pure steel, whereas the bottom surface of the beam is pure silicon nitride. The material properties of the constituents are given in (Librescu and Al. 2008). The parameters of the beam are $L = 1$ m and $\lambda = 70$.

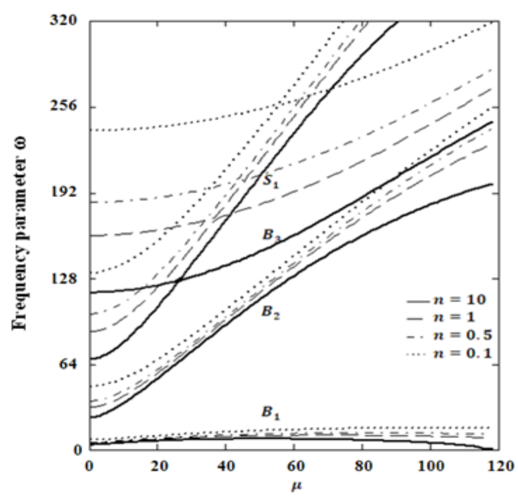

Fig. 3 Variation of frequency parameters of stainless steel-silicon nitride RFGB

Fig. 3 show how the frequency parameter of the first three bending modes and the first stretching mode of the rotating beam vary with rotating speed parameter and the power-law exponent. The bending frequency curves B2-B3 increase with increasing rotating speed parameter. The rate of increase depends on the power-low exponent, therefore of the beam rigidity. Under the effect of gyroscopic coupling, the first bending curves B1 for n = 10 decreases to a value of zero; the rotating beam will buckle at zero natural frequencies. This natural frequency is called buckling speed where is equal to 117.39. For other values of n (< 10), the beam becomes more rigid, therefore less sensitive to the softening effect caused by the gyroscopic coupling. The values of the buckling speed, in this case are largely superior to 117.39. On the other hand, the stretching curves S1 increases with increasing rotating speed parameter.

Another interesting phenomenon can be observed in Fig. 3 is called veering modes. The third bending frequency curve B3 and the first stretching frequency curve S1 veer at $\mu = 27.06$ for n=10. The mode shape changes abruptly around the veering region (Yoo, and Shin 1998). The same phenomenon occurs for other values of n.

6. Conclusion

In this paper, transverse vibrations of the lead-lag motion of a uniform rotating beam with functionally graded materials have been investigated. Using the stretch deformation instead of the conventional axial deformation, linear partial deferential equations of motion of a RFGB have been derived by Hamilton's principle; in which the coupling between stretching and bending motions is considered. These equations include the combined effects of the centrifugal stiffening force, the centripetal acceleration and the gyroscopic force. A two node beam Fourier p-element has been developed and used with 4-dof at each node: transverse displacement, the slope and the curvature and the stretching displacement in the discretization of the weak forms of the partial differential equations of motion.
The new element alleviates the convergence difficulties caused by the effect of centrifugal stiffening which strongly influences the vibrational behavior. Superior convergence is found to occur as the number of trigonometric hierarchical functions is increased compared to the conventional finite elements. Accurate results were obtained using only one element, it is shown that by this element the order of the resulting matrices in the FEM is considerably reduced leading to a significant decrease in computational effort.

Numerical results show that the natural frequencies and modes shape of the RFGB are influenced significantly by varying individually or jointly the angular speed, Young’s modulus ratio and power-law exponent.

7. Reference

[1] M. KoiZumi, “FGM activities in Japan”, *Composites B*, **28** 1–4, 1997.

[2] Y. Miyamoto, W.A. Kaysser, B.H. Rabin, A. Kawasaki, and R.G. Ford, “Functionally Graded Materials: Design, Processing and applications”, *Kluwer Academic Publishers*, 1999.

[3] S.Y. Oh, , L. Librescu, and O. Song, “Thermoelasticmodeling and vibration of functionally graded thin–walled rotating blades”, *AIAA J.*, **41**, 2051–2060, 2003.

[4] L. Librescu, , S.Y. Oh, and O. Song, “Thin–walled beams made of functionally graded materials and operating in a high temperature environment: vibration and stability”, *J. of Therm. Stress.*,**28**, 649–712, 2005.

[5] L. Librescu, S.Y. Oh, O. Song, and H.S. Kang, “Dynamics of advanced rotating blades made of functionally graded materials and operating in a high-temperature field”, *J. Eng. Math*,**61**, 1–16, 2008.

[6] S.A. Fazelzadeh, and M. Hosseini, “Aerothermoelastic behavior of supersonic rotating thin–walled beams made of functionally graded materials”, *J. Fluids Struct.*, **23**, 1251–1264, 2007.

[7] M.T. Piovan, and R. Sampaio, “A study on the dynamics of rotating beams with functionally graded properties”, *J. Sound Vib*.,**327**, 134–143, 2009.

[8] J. `rotating cantilever beam by using the finite element method”, *J. Sound Vib.*,**249**, 147–164, 2002.

[9] S.M. Hamza-Cherif, “Free vibration analysis of rotating flexible beams by using the Fourier p-version of the finite element method”, *Int. J. Comput., Methods*,**2**, 255-269, 2005.